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ON THE COHOMOLOGY GROUPS OF CERTAIN COVERING SPACES

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0. Introduction

Deformation Theory of compact complex manifolds has been studied by many people and they have obtained important results. Recently, T.Ohsawa has studied the stability of a family of Riemann surfaces and proved the following by applying results of Teichimüller Theory(cf:[Oh2]): Let X be a connected complex manifold of dimension 2 and U be the unit disk of $\mathbf{C}$. Let $\pi : X \longrightarrow U$ be a proper surjective holomorphic map with maximal rank. Then every covering space of X is holomorphically convex.

In this paper we consider a higher dimensional version of this Theorem as the following: Let X be a complex manifold of dimension $N = n+m$ and T be a complex manifold of dimension m,where n and m are positive integers. Let $\pi : X \longrightarrow T$ be a proper surjective holomorphic map with maximal rank. Let $\sigma : \widetilde{X} \longrightarrow X$ be a covering map. $\widetilde{X}_A$ denotes $(\pi \circ \sigma)^{-1}(A)$ for $A \subset T$. $H^q(X, \mathcal{F})$ denotes the sheaf cohomology group of X of degree q with coefficients $\mathcal{F}$,where $\mathcal{F}$ denotes a coherent analytic sheaf over X. Then we have the following Theorem.

Theorem. *Suppose that each fiber of $\pi \circ \sigma$ is non compact. Then each point of T has a neighborhood U satisfying $H^i(\widetilde{X}_U, \mathcal{F}) = 0$ for $n \leq i \leq N$,where $\mathcal{F}$ is any coherent analytic sheaf over $\widetilde{X}_U$.*

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If each fiber of $\pi \circ \sigma$ is non compact and connected, there is a strongly n–convex exhaustion function on each fiber of $\pi \circ \sigma$(cf:[G–W 2]).Hence we have $H^n(\widetilde{X}_z, \mathcal{F}_z) = 0$ at $z \in T$ for any coherent analytic sheaf $\mathcal{F}_z$ over $\widetilde{X}_z$ by Theorem of Andreotti–Grauert(cf:[A–G]). Our Theorem claims that 'Union problem' is solved on a sufficiently small neighborhood U of $z \in T$ with respect to Cohomology vanishings.

To show our claim we examine Theorem of Kuranishi precisely(cf:[Ku]). On the base of the results , we construct a Morse function with convexity properties. We have our claim by making use of the function. Homology Theory has been studied by handling real–analytic Morse functions (cf :[Vâ]). However, investigations of cohomology groups by applying Morse functions do not have been done so much.

The organization of this paper is as the followings. In §1 we explain properties of strongly q–complete manifolds and an abstract vanishing Theorem. In §2 we introduce results of Kuranishi and show existences of good C^∞–maps on $\widetilde{X}_U$. In §3 we show existences of the particular Morse function on $\widetilde{X}_0$. Then we prove convexity properties of some relatively compact domains of $\widetilde{X}_U$. In §4 we construct an exhaustive sequence of Runge pairs ,which is an alternative of Docquier–Grauert's argument (cf:[D–G]), and show our claim.

1. Preliminaries

Let X be a complex manifold of dimension n and let q be an integer with $1 \leq q \leq n$. $T_x^{1,0}X$ denotes the holomorphic tangent space of X at $x \in X$ and $T^{1,0}X$ denotes the holomorphic tangent bundle of X. A real–valued C^2–function φ on X is said to be strongly q–convex at a point $x \in X$ if its Levi form of φ has at least $n - q + 1$ positive eigenvalues on $T_x^{1,0}X$ at x. The function φ is said to be strongly q–convex on X if it is strongly q–convex at any point of X.

A real–valued function φ on X is said to be an exhaustion function if the sublevel set $X_c := \{p \in X | \varphi(p) < c\}$ is relatively compact for any $c \in \mathbf{R}$.

A complex manifold X is said to be strongly q–convex if there exists a compact subset K of X and an exhaustion function φ ,which is strongly q–convex on $X \setminus K$. If we can choose $K = \phi$, X is said to be strongly q–complete.

Let $\mathcal{U} := \{U_i\}_{i \in N}$ be a countable Stein open covering of a complex manifold X such that $\mathcal{U}$ is a base of open sets for the topology of X.If $\mathcal{F}$ is a coherent analytic sheaf over X ,we denote by $C^p(\mathcal{U}, \mathcal{F})$ the Fréchet space of Cěch cochains , $\delta = \delta_p$: $C^p(\mathcal{U}, \mathcal{F}) \longrightarrow C^{p+1}(\mathcal{U}, \mathcal{F})$ the coboundary operator, $Z^p(\mathcal{U}, \mathcal{F}) := \ker \delta_p$ the Fréchet space of cocycles. If $D \subset X$ is an open set , we define $\mathcal{U}|_D := \{U_i \in \mathcal{U} | U_i \subset D\}$.

Theorem1.1(cf:[A–G]). *Let X be a strongly q–complete manifold and $\mathcal{F}$ be any coherent analytic sheaf over X and let $\mathcal{U}$ be as above. Then the followings hold. (1) The restriction map $Z^i(\mathcal{U}, \mathcal{F}) \longrightarrow Z^i(\mathcal{U}|_{X_c}, \mathcal{F})$ has dense image for any $c \in \mathbf{R}$ if $i \geq q$ (2) $H^i(X, \mathcal{F}) = 0$ holds if $i \geq q$.*

Proof: See [A–G].

Let X be a complex manifold. Let $\{G_k \subset X\}_{k \in \mathbf{N}}$ be a sequence of strongly q–complete open subsets such that $\Psi_k : G_k \longrightarrow \mathbf{R}$ is a strongly q–convex exhaustion function on G_k for $k \in \mathbf{N}$. We say that $\{(G_k, \Psi_k)\}_{k \in \mathbf{N}}$ is **an exhaustion sequence of q–Runge pairs** on X if there is a sequence of set $\{M_k \subset G_k\}_{k \in \mathbf{N}}$ and a sequence of numbers $\{C_k \in \mathbf{R}\}_{k \in \mathbf{N}}$ satisfying followings: (i) $G_k \subset L_k := \{p \in G_{k+1} | \Psi_{k+1}(p) < C_k\}$ holds and M_k is a compact subset of L_k (ii) $X = \bigcup_{k \in \mathbf{N}} M_k$.

Then we have the following. It is intrinsic Proposition when we show our claim.

Proposition1.2(cf:[Si]). *Let X be a complex manifold and $\mathcal{F}$ be any coherent analytic sheaf over X. Let $\{G_k \subset X\}_{k \in \mathbf{N}}$ be a sequence of strongly q–complete open subsets such that $\Psi_k : G_k \longrightarrow \mathbf{R}$ is a strongly q–convex exhaustion function on G_k for $k \in \mathbf{N}$. Suppose that $\{(G_k, \Psi_k)\}_{k \in \mathbf{N}}$ is an exhaustion sequence of q–Runge pairs on X. Then we have $H^i(X, \mathcal{F}) = 0$ if $i \geq q$.*

Proof: We have Proposition1.2 from Theorem1.1 and the argument to show Theorem B of Cartan.

2.Complex analytic families of complex structures of a compact manifold

From now on , let X be a complex manifold of dimension $N = n + m$ and T be a complex manifold of dimension m. Let $\pi : X \longrightarrow T$ be a proper surjective holomorphic map with maximal rank. We put $X_A := \pi^{-1}(A)$ for $A \subset T$. We may assume that T is a domain of $\mathbf{C}^m$ which contains $0 \in \mathbf{C}^m$. Let $\{h_\mu : \mathcal{U}_\mu \longrightarrow V_\mu \times U\}_{\mu=1,\cdots,k}$ be a local coordinate system of X_U ,where $\{\mathcal{U}_\mu\}$ are open subsets of X_U and $\{V_\mu\}$ are open subsets of $\mathbf{C}^n$ and U is an open subset of T which contains $0 \in \mathbf{C}^m$. Moreover we suppose that there is a point $p_\mu \in V_\mu$ for any $\mu = 1, \cdots, k$ such that p_μ is not contained in V_ν for any $\mu \neq \nu$. $\mathcal{X}_0$ denotes the underlying C^∞–manifold of X_0. We remark that U will be replaced with sufficiently small one ,and $\{V_\mu\}$ is regarded as an open covering of $\mathcal{X}_0$ or X_0 ,whenever necessary. Then followings holds. It is an exact observation for results of Kuranishi.

Theorem2.1(cf:[Ku]p.26,Theorem3.2). *There are a neighborhood U of $z \in T$ and a diffeomorphism $S : \mathcal{X}_0 \times U \ni (x, z) \mapsto S(x, z) \in X$ satisfying the followings: (i) $S(\mathcal{X}_0, z) = X_z$ for any $z \in U$ (ii) $U \ni z \mapsto S(x, z) \in X$ is a holomorphic section over U for any $x \in \mathcal{X}_0$ and X_U is the disjoint union of $\{S(x, U) | x \in \mathcal{X}_0\}$ (iii) $r : X_U \ni p \mapsto r(p) \in X_0$ defined by $S(r(p), \pi(p)) = p$ is a C^∞–retraction ,where we identify $\mathcal{X}_0 \times \{0\}$ with X_0 (iv) there is a neighborhood $W_\mu \subset\subset V_\mu$ of p_μ such that r is a holomorphic retraction from $S(W_\mu, U)$ to $W_\mu \subset X_0$ for any $\mu = 1, \cdots, k$.*

Proof: See [Ku] for the detail of Proof. Here we give an observation for (iv). There is a diffeomorphism $G : X_U \longrightarrow \mathcal{X}_U \times U$. We can construct G by patching maps in local coordinates. Further there is a neighborhood $\mathcal{W} \subset X_0 \times X_U$ of the diagonal set $\{(x, x) | x \in X_0\}$, a diffeomorphism $F : \mathcal{W} \longrightarrow F(\mathcal{W}) \subset T^{1,0}X_0 \times U$ such that (A) $F(\mathcal{W}_x) \subset T^{1,0}_x X_0 \times U$ (B) $F(x, z) = (0, \pi(z)) \in T^{1,0}_x X_0 \times U$ (C) $F|_{\mathcal{W}_x}$ is a biholomorphic for fixed $x \in X_0$, where we put $\mathcal{W}_x := (\{x\} \times X_U) \cap \mathcal{W}$. Existences of F in local coordinates is trivial. Hence we can also construct F by patching them.We can define S satisfying Theorem2.1 by the use of G and F. Especially we construct S satisfying (iv) since S is defined by using G and F ,which consist of patching maps in local coordinates.

Remark2.2 Theorem 2.1 claims the existence of 'holomorphic motion' for any dimensional fibers and any dimensional base spaces on a sufficiently small neighborhood of $z \in T$. If $n = m = 1$ and T is the unit disk in $\mathbf{C}$,it has been shown that we can take $U = T = \{z \in \mathbf{C} || z| < 1\}$ by using Teichimüller theory.

Let $\pi : X \longrightarrow T$ be as above. Let $\sigma : \widetilde{X} \longrightarrow X$ be any covering map. $\widetilde{X}_A$ denotes $(\pi \circ \sigma)^{-1}(A)$ for $A \subset T$. We fix a local coordinate system $\{V_\mu\}_{\mu=1,\cdots,k}$ of X_0 such that V_μ and each connected component of $\sigma^{-1}(V_\mu)$ are biholomorphic to the unit disk of $\mathbf{C}^n$. We suppose that there is a point $p_\mu \in V_\mu$ for any $\mu = 1, \cdots, k$ such that p_μ is not contained in V_ν for any $\mu \neq \nu$. We set $U := \{z \in \mathbf{C}^m || z| < 1\}$. We may suppose that there are C^∞–maps $S : \mathcal{X}_0 \times U \longrightarrow X_U$ and $r : X_U \longrightarrow X_0$ satisfying Theorem2.1 for the local coordinate system $\{V_\mu\}_{\mu=1,\cdots,k}$ by altering the coordinate of T if necessary. We fix such C^∞–maps S and r. W_μ denotes the open subset of V_μ satisfying Theorem2.1 (iv). $\{\widetilde{V}_{\mu,\alpha} | \mu = 1, \cdots, k, \alpha \in A\}$ denotes the connected components of $\sigma^{-1}(V_\mu)$. Then $\{\widetilde{V}_{\mu,\alpha}\}$ is a locally open covering of $\widetilde{X}_0$. Let $\{j = j(\mu, \alpha) \in \mathbf{N}\}$ be the set of another indices which corresponds to the set of pairs of indices $\{(\mu, \alpha) | \mu = 1, \cdots, k, \alpha \in A\}$. We denote by $\{\widetilde{V}_j\}$ the open covering $\{\widetilde{V}_{\mu,\alpha}\}$, where $j = j(\mu, \alpha)$ for $\mu = 1, \cdots, k$ and $\alpha \in A$. Each lift of S and r to $\widetilde{X}$ is well–defined since S satisfies Theorem2.1(ii). We denote the lift of S to $\widetilde{X}$ by $\widetilde{S} : \widetilde{\mathcal{X}}_0 \times U \longrightarrow \widetilde{X}_U$ and the lift of r to $\widetilde{X}$ by $\widetilde{r} : \widetilde{X}_U \longrightarrow \widetilde{X}_0$. $\widetilde{W}_j$ denotes the connected component of $\sigma^{-1}(W_\mu)$ contained in $\widetilde{V}_j$,where $j = j(\mu, \alpha)$ for $\mu = 1, \cdots, k$ and $\alpha \in A$. $\widetilde{\mathcal{X}}_0$ denotes the underlying C^∞–manifold of $\widetilde{X}_0$. We regard $\{\widetilde{V}_j\}$ as an open covering of $\widetilde{\mathcal{X}}_0$ or $\widetilde{X}_0$ whenever necessary. Then the following holds from Theorem2.1.

Theorem2.3. *$\widetilde{S} : \widetilde{\mathcal{X}}_0 \times U \ni (y, z) \mapsto \widetilde{S}(y, z) \in \widetilde{X}$ is a diffeomorphism satisfying the followings: (i) $\widetilde{S}(\widetilde{\mathcal{X}}_0, z) = \widetilde{X}_z$ holds for any $z \in U$ (ii) $U \ni z \mapsto \widetilde{S}(y, z) \in \widetilde{X}$ is a holomorphic section over U for any $y \in \widetilde{\mathcal{X}}_0$ and $\widetilde{X}_U$ is the disjoint union of $\{\widetilde{S}(y, U) | y \in \widetilde{\mathcal{X}}_0\}$ (iii) $\widetilde{r} : \widetilde{X}_U \ni p \mapsto \widetilde{r}(p) \in \widetilde{X}_0$ satisfies $\widetilde{S}(\widetilde{r}(p), \pi(p)) = p$ and $\widetilde{r}$ is a C^∞–retraction (iv) $\widetilde{r}$ is a holomorphic retraction from $\widetilde{S}(\widetilde{W}_j, U)$ to $\widetilde{W}_j \subset \widetilde{X}_0$ for any j.*

3.Convexity of some relatively compact domains of $\widetilde{X}$

At the beginning, we introduce the followings in accordance with Demailly (cf:[De], [G–W2],[Oh1]) : Let M be an n–dimensional complex manifold with a hermitian metric $g = (g_{i\bar{j}})_{1\leq i,j\leq n}$.For a C^2–function v, we consider the trace of the Levi form with respect to g defined by

$$\triangle_g v(p) = Trace_g\sqrt{-1}\partial\bar{\partial}v(p) := \sum_{1\leq i,j\leq n} \sqrt{-1}g^{i\bar{j}}(p)\frac{\partial^2 v}{\partial z_i \partial \overline{z_j}}(p)$$

,where $(g^{i\bar{j}})$ is the inverse matrix of $(g_{i\bar{j}})$. Then $\triangle_g v$ is a continuous function on M. We will say that v is strongly g–subharmonic if $\triangle_g v(p) > 0$ for any $p \in M$. v is strongly n–convex if v is strongly g–subharmonic. If v_1 and v_2 is strongly g–subharmonic, $v_1 + v_2$ is strongly g–subharmonic. Let N be a complex submanifold of M and $v : N \longrightarrow \mathbf{R}$ be a C^2–function. We set $\triangle_g v|_N(p) := \sum g^{i\bar{j}}|_N(p) \cdot \frac{\partial^2 v}{\partial z_i \partial \overline{z_j}}|_N(p)$ for any $p \in N$,where $(g^{i\bar{j}}|_N(p))$ denotes the inverse matrix of $(g_{i\bar{j}}(p))$ which is restricted on the cotangent space of N at $p \in N$ and $(\frac{\partial^2 v}{\partial z_i \partial \overline{z_j}}|_N(p))$ denotes the restriction of $(\frac{\partial^2 v}{\partial z_i \partial \overline{z_j}})$ on the tangent space of N at p. We will say that v is strongly g–subharmonic on N if $\triangle_g v|_N(p) > 0$ for any $p \in N$.

Theorem3.1(cf:[De],Theorem9). *Every n–dimensional connected non compact complex manifold has a strongly subharmonic exhaustion function with respect to any hermitian metric g.*

Proof: See [De].

On the other hand, we can approximate q–convex functions by real–analytic q–convex Morse functions as the following.

Theorem3.2(cf:[Vâ],Corollary5). *Let M be a q–complete manifold and $\varphi : M \longrightarrow \mathbf{R}$ a q–convex exhaustion function . Then for any continuous function $\varepsilon : M \longrightarrow (0,\infty)$ there is $\psi : M \longrightarrow \mathbf{R}$ such that (i)ψ is real–analytic and q–convex (ii)ψ is a Morse function. Hence ψ has distinct critical values and the set of critical points is discrete in M (iii)$|\psi - \varphi| < \varepsilon$.*

Proof: See [Vâ].

Now let us return to the original situation which we have observed in the previous section. Let $\pi : X \longrightarrow T$ be as above. Let $\sigma : \widetilde{X} \longrightarrow X$ be a covering map such that each fiber of $\pi \circ \sigma$ is non compact. Let g be a hermitian metric on $\widetilde{X}$. We may assume that each fiber of $\pi \circ \sigma$ is connected.

Proposition3.3(cf:[Oh2]). *Let $G \subset\subset \widetilde{X}$ be any relatively compact open subset . Then there is a C^∞–function $\widetilde{\varphi}$ on G such that $\widetilde{\varphi}$ is strongly g–subharmonic on each n–dimensional complex submanifold $G \cap \widetilde{X}_z$ for any $z \in (\pi \circ \sigma)(G)$.*

Proof: Let $\{U_i \subset T\}_{i=1,\cdots,l}$ be a finite open covering of $(\pi \circ \sigma)(\overline{G})$ satisfying that there are a finite open covering $\{V_i \subset \widetilde{X}\}_{i=1,\cdots,l}$ of $\overline{G}$ with $V_i \supset G \cap \widetilde{X}_{U_i}$ and bounded C^∞ functions ψ_i on V_i for any $i = 1, \cdots, l$ such that $\psi_i|_{V_i \cap \widetilde{X}_z}$ is strongly g–subharmonic on $V_i \cap \widetilde{X}_z$ for any $z \in \overline{U}_i$. Let $\{\rho_i\}$ be a partition of unity subordinate to $\{U_i\}$.We set $\widetilde{\varphi}(p) := \sum \rho_i((\pi \circ \sigma)(p))\psi_i(p)$ for any $p \in G$. Then $\widetilde{\varphi}|_{G \cap \widetilde{X}_z}$ is strongly g–subharmonic on $G \cap \widetilde{X}_z$ for any $z \in (\pi \circ \sigma)(G)$.

Let $\widetilde{S} : \widetilde{\mathcal{X}}_0 \times U \longrightarrow \widetilde{X}_U$, $\widetilde{r} : \widetilde{X}_U \longrightarrow \widetilde{X}_0$, $\{\widetilde{V}_j\}$, $\{\widetilde{W}_j\}$ be the same as these in the previous section. Then we have the following.

Proposition3.4. *There is a Morse exhaustion function h on $\widetilde{X}_0$ satisfying that (i) h has distinct critical values (ii) the set of critical points of h is contained in $\bigcup_{j \in \mathbf{N}} \widetilde{W}_j$ (iii) h is strongly n–convex on $\bigcup_{j \in \mathbf{N}} \widetilde{W}_j$.*

Proof: By Theorem3.1 and Theorem3.2 there is a strongly n–convex Morse exhaustion function h_0 on $\widetilde{X}_0$. Let $\{y_i^*\}_{i \in \mathbf{N}}$ be the set of critical points of h_0. Then we construct a map $j : \mathbf{N} \ni i \mapsto j(i) \in \mathbf{N}$ such that $y_i^* \in \widetilde{V}_{j(i)}$. The map is not determined uniquely. Hence we fix such a map j. Then $\#\{y_i^* | y_i^* \in \widetilde{V}_k, k = j(i)\}$ is finite for any $k \in \mathbf{N}$ since $\{y_i^*\}$ is the discrete set. By Theorem2.1 (iv) there is an open subset $\widetilde{W}_{j(i)} \subset \widetilde{V}_{j(i)}$ such that $\widetilde{r}|_{\widetilde{S}(\widetilde{W}_{j(i)}, U)}$ is the holomorphic retraction for $i \in \mathbf{N}$. Let $N_i^* \in \widetilde{X}_0$ be an open neighborhood of y_i^* which is contained in $\widetilde{V}_{j(i)}$. Let $N_i \subset \widetilde{W}_{j(i)}$ be an open subset which is biholomorphic to N_i^*. Further we suppose that the sequence $\{N_i\}_{i \in \mathbf{N}}$ does not have intersections each other.

Let $\{f_k\}_{k \in \mathbf{N}}$ be a sequence of diffeomorphisms from $\widetilde{X}_0$ to oneself satisfying that f_k is a biholomorphic map from N_i^* to N_i for any $j(i) \leq k$, and $f_k = f_{k-1}$

on $\big(\bigcup_{l\geq k} \widetilde{V}_l\big)^c$. Such a sequence $\{f_k\}$ exists since $\#\{y_i^*|y_i^* \in \widetilde{V}_k, k = j(i)\}$ is finite for $k \in \mathbf{N}$ and $\widetilde{W}_k \cap \widetilde{W}_l$ are empty for $k \neq l$. We set $f := \lim_{k \longrightarrow \infty} f_k$. f is a diffeomorphism from $\widetilde{X}_0$ to oneself such that f is biholomorphic from N_i^* to N_i for $i \in \mathbf{N}$. We put $h := h_0 \circ f^{-1}$ and $y_i := f(y_i^*)$. Then h is a Morse exhaustion function on $\widetilde{X}_0$ and the set of critical points $\{y_i\}_{i\in\mathbf{N}}$ is contained in $\bigcup_{j\in\mathbf{N}} \widetilde{W}_j$. Further h is strongly n–convex on $\bigcup_{j\in\mathbf{N}} \widetilde{W}_j$ since φ is biholomorphic on $\bigcup_{j\in\mathbf{N}} \widetilde{W}_j$. Thus we have Proposition3.4.

We fix the Morse exhaustion function h satisfying Proposition3.4. We may assume that $h(\widetilde{X}_0) = [0, \infty)$, by replacing h with $\lambda \circ h$ for an unbounded strictly increasing convex function $\lambda : [\inf h, \sup h) \longrightarrow [0, \infty)$, if necessary. We put $D(t) := \{y \in \widetilde{X}_0 | h(y) < t\}$, $U(t) := \{z \in \mathbf{C}^m | -\log(1 - |z|^2) < t\}$ for $t \in [0, \infty)$. We put $\widetilde{S}(A, B) := \bigcup_{y\in A, z\in B} \widetilde{S}(y, z)$ for $A \subset \widetilde{X}_0$ and $B \subset U$.

$h \circ \widetilde{r}$ is strongly n–convex on each n–dimensional manifold $\widetilde{S}(\widetilde{W}_j, U) \cap \widetilde{X}_z$ for $j \in \mathbf{N}$ at $z \in U$ by Proposition3.4. We fix a hermitian metric g on $\widetilde{X}_U$ such that $h \circ \widetilde{r}$ is strongly g–subharmonic on $\widetilde{S}(\widetilde{W}_j, U) \cap \widetilde{X}_z$ for $j \in \mathbf{N}$ at $z \in U$ (cf:[De],Lemma6).

Let M, N be hermitian forms on a vector space L. We say that M is positive (resp. non negative) definite if each eigenvalue of M is positive (resp.non negative). $M > 0$ (resp.$M \geq 0$) means that M is positive (resp. non negative) definite. $M > N$(resp.$M \geq N$) means that $M - N$ is positive (resp.non negative) definite.We use same notations for hermitian matrices. We denote by $|M|$ the determinant of the hermitian matrix M.

Then we have the following Proposition with respect to the C^∞–map $\widetilde{S}$ and the Morse function h ,which have been fixed in our argument.

Proposition3.5. *$\widetilde{S}(D(t), U(t))$ is strongly n–complete for any $t \in (0, \infty)$.*

Proof: We set $G(t) := \widetilde{S}(D(t), U(t))$. We put $D(s, t) := \{p \in \widetilde{X}_0 | s \leq h(p) < t\}$, $G(s, t) := \widetilde{S}(D(s, t), U(t))$. We put $\widetilde{h}(p) := (h \circ \widetilde{r})(p)$, $\mu(p) := -\log(t - \widetilde{h}(p))$ for $p \in \widetilde{S}(D(t), U)$, $\nu(p) := -\log(1 + \exp(-t) - |(\pi \circ \sigma)(p)|^2)$ for $p \in \widetilde{S}(\widetilde{X}_0, U(t))$.

We put $Z(p) := \{\widetilde{S}(y, z) | y \in \widetilde{X}_0$ satisfying that $\widetilde{S}(y, (\pi \circ \sigma)(p)) = p$, $z \in U\}$

for $p \in \widetilde{X}_U$. $Z(p)$ is an m–dimensional complex submanifold of $\widetilde{X}_U$,which is biholomorphic to U from Theorem2.5(ii). On the other hand , $Y(z) := G(t) \cap \widetilde{X}_z$ is an n–dimensional complex submanifold of $G(t)$ for $z \in U(t)$. We fix a constant $t^* \in (t, \infty)$. Let $\widetilde{\varphi}$ be a bounded C^∞–function on $G(t^*)$ such that $\widetilde{\varphi}$ is strongly g–subharmonic on $G(t^*) \cap \widetilde{X}_z$ at $z \in U(t^*)$ in Proposition3.3. We put $\Psi(p) := \mu(p) + \nu(p) + A|(\pi \circ \sigma)(p)|^2 + B\widetilde{\varphi}(p)$ for $p \in G(t)$,where A and B are positive constants.

Case 1: Let $t \in (0, \infty)$ be a regular value of h. Then $\triangle_g \mu|_{Y((\pi\circ\sigma)(p))}$ is bounded from below on $G(t)$. Indeed, let $\{p_i \in Y(z)\}$ be a sequence of points such that $\{p_i\}$ converges to a point $p_0 \in \widetilde{S}(\partial D(t), z)$ for $z \in U(t)$. Then we have

$\triangle_g \mu|_{Y(z)}(p_i) = Trace_g \sqrt{-1}\left(\frac{\partial\widetilde{h}\wedge\bar{\partial}\widetilde{h}}{(c-\widetilde{h})^2} + \frac{\partial\bar{\partial}\widetilde{h}}{c-\widetilde{h}} \right)\Big|_{Y(z)} (p_i) \longrightarrow \infty$ as $i \longrightarrow \infty$, since $\partial\widetilde{h} \neq 0$ holds on $\widetilde{S}(\partial D(t), U)$. Hence Ψ is strongly g–subharmonic on each $Y(z)$ for $z \in U(t)$ if B is sufficiently large. We fix such a positive constant B.

Let $b \in (0, \infty)$ be a number such that h does not have critical points on $D(b, t)$. Let p be any point of $G(b, t)$. Let $l(p) \subset T^{1,0}_p Y((\pi\circ\sigma)(p))$ be the 1–dimensional complex subspace containing $(1, 0)$–part of the normal vector of the level surface $\{\widetilde{h}(p) = s\}$ in $\widetilde{X}_U$ for $s \in [b, t)$. We put an $(m + 1)$–dimensional complex subspace $H(p) := T^{1,0}_p Z(p) \oplus l(p) \subset T^{1,0}_p G(t)$. Then we have $\sqrt{-1}\partial\bar{\partial}\Psi \geq \sqrt{-1}\partial\bar{\partial}\mu + \sqrt{-1}\partial\bar{\partial}(A \sum |(\pi \circ \sigma)|^2) + \sqrt{-1}\partial\bar{\partial}(B\widetilde{\varphi})$ on $T^{1,0}_p G(t)$. Let $\tau(p) = (\tau_1(p), \cdots, \tau_{m+1}(p))$ be a normal basis of $H(p)$ with respect to g satisfying that $span\langle\tau_1(p), \cdots, \tau_m(p)\rangle_{\mathbf{C}} = T^{1,0}_p Z(p)$,$span\langle\tau_{m+1}(p)\rangle_{\mathbf{C}} = l(p)$ for $p \in G(b, t)$ and τ_i are continuous sections in $T^{1,0}G(b, t)$ for $i = 1, \cdots, m + 1$. Then we have

$$\sqrt{-1}\partial\bar{\partial}\mu =: \begin{pmatrix} 0 & \cdots & 0 & \frac{b_1}{(t-\widetilde{h})} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \frac{b_m}{(t-\widetilde{h})} \\ \frac{\bar{b}_1}{(t-\widetilde{h})} & \cdots & \frac{\bar{b}_m}{(t-\widetilde{h})} & \frac{b_0}{(t-\widetilde{h})^2} + \frac{b_{m+1}}{(t-\widetilde{h})} \end{pmatrix}$$

as the matrix representation with respect to τ, where b_0 is a continuous function such that there is a constant $a_0 > 0$ satisfying that $b_0(p) > a_0$ for any $p \in G(b, t)$ and b_i are bounded functions on $G(b, t)$ for $i = 1, \cdots, m + 1$. We put $\sqrt{-1}\partial\bar{\partial}(A \sum |(\pi \circ$

$\sigma)|^2) =: A(c_{ij})_{1\leq i,j\leq m+1}$ with respect to τ, where c_{ij} are bounded functions on $G(b,t)$. We put $C_1 := (c_{ij})_{1\leq i,j\leq m}$. Then there is a constant $a_1 > 0$ satisfying that $C_1 > a_1 I_m$ on $G(b,t)$,where $I_m(p)$ denotes the identity matrix, since $(\pi\circ\sigma)(Z(p)) = U$ holds for $p \in \widetilde{X}_U$. We set $\sqrt{-1}\partial\overline{\partial}B\widetilde{\varphi} =: (d_{ij})_{1\leq i,j\leq m+1}$ with respect to τ, where d_{ij} are bounded functions on $G(b,t)$. We put $C_2 := (d_{ij})_{1\leq i,j\leq m}$. Each element of C_1 and C_2 is bounded on $G(b,t)$. So we have $AC_1(p) + C_2(p) > 0$ for $p \in G(b,t)$ if A is sufficiently large. M denotes the matrix representation of the hermitian form $\sqrt{-1}\partial\overline{\partial}\mu + \sqrt{-1}\partial\overline{\partial}(A\sum|(\pi\circ\sigma)|^2) + \sqrt{-1}\partial\overline{\partial}B\widetilde{\varphi}$ with respect to τ. Then we have $|M| = \frac{1}{(t-\widetilde{h})^2}\{b_0A^m|C_1| + Q_{m-1}(A)\} + \frac{1}{t-\widetilde{h}}Q_m(A) + Q_{m+1}(A)$,where $Q_i(A)$ denotes the i–degree polynomial with respect to A whose coefficients are bounded functions on $G(b,t)$. Let $\{q_i \in G(t)\}$ be a sequence such that $\{q_i\}$ converges to a point $q_0 \in \widetilde{S}(\partial D(t), \overline{U(t)})$. Then we have $|M(q_i)| \longrightarrow +\infty$ as $i \longrightarrow +\infty$ if A is sufficiently large. Hence there is a constant $c \in (b,t)$ and a sufficiently large constant A satisfying that $|M(p)| > 0$ for any $p \in G(c,t)$. Then we have $\sqrt{-1}\partial\overline{\partial}\Psi(p) > 0$ on $H(p)$ for any $p \in G(c,t)$ if A is sufficiently large. We fix such a constant $A =: A_1$.

Let $d \in (c,t)$ be a number. Ψ is strongly g–subharmonic on each $Y(z)$ for any $z \in U(t)$ on $G(t)$. Hence there is an 1–dimensional complex subspace $\hat{l}(p) \subset T^{1,0}_pY((\pi\circ\sigma)(p))$ satisfying that $\partial\overline{\partial}\Psi|_{\hat{l}(p)} > 0$ for $p \in G(t)$. Let p be any point of $G(t)\setminus G(c,t)$. Let $\{(z_1,\cdots,z_m,w_{n+1},\cdots,w_{n+m}), U_p\}$ be a local coordinate around $p \in G$ such that $(z_1(q),\cdots,z_m(q)) \in \mathbf{C}^m$ is a coordinate of $(\pi\circ\sigma)(q) \in T$ for $q \in T$. We put an $(m+1)$–dimensional complex subspace $\hat{H}(p) := span\langle\frac{\partial}{\partial z_1}(p),\cdots,\frac{\partial}{\partial z_m}(p)\rangle_{\mathbf{C}} \oplus \hat{l}(p) \subset T^{1,0}_pG(d)$. Then we have $\partial\overline{\partial}\Psi(p) > 0$ on $\hat{H}(p)$ for $p \in G(d)$ if A is sufficiently large. Indeed followings holds. Let $\hat{\tau}(p) = (\hat{\tau}_1(p),\cdots,\hat{\tau}_{m+1}(p))$ be a normal basis of $\hat{H}(p)$ with respect to g satisfying that $span\langle\hat{\tau}_1(p),\cdots,\hat{\tau}_m(p)\rangle_{\mathbf{C}} = span\langle\frac{\partial}{\partial z_1}(p),\cdots,\frac{\partial}{\partial z_m}(p)\rangle_{\mathbf{C}}$,$span\langle\hat{\tau}_{m+1}(p)\rangle_{\mathbf{C}} = \hat{l}(p)$ and $\hat{\tau}_i$ are continuous sections in $T^{1,0}G(d)$ for $i = 1,\cdots,m+1$. Then we have $\sqrt{-1}\partial\overline{\partial}\mu =: (\hat{b}_{ij})_{1\leq i,j\leq m+1}$ with respect to $\hat{\tau}$, where $\hat{b}_i$ are bounded functions on $G(d)$. We put $\sqrt{-1}\partial\overline{\partial}(A\sum|(\pi\circ\sigma)|^2) =: A\begin{pmatrix}\hat{C}_1 & 0\\ 0 & 0\end{pmatrix}$ with respect to τ. Then there is a constant $a_2 > 0$ satisfying that $\hat{C}_1 > a_2 I_m$ since $span\langle\hat{\tau}_1(p),\cdots,\hat{\tau}_m(p)\rangle_{\mathbf{C}} =$

$span\langle \frac{\partial}{\partial z_1}(p), \cdots, \frac{\partial}{\partial z_m}(p)\rangle_{\mathbf{C}}$ holds. We set $\sqrt{-1}\partial\overline{\partial}B\widetilde{\varphi} =: (\hat{d}_{ij})_{1\leq i,j\leq m+1}$ with respect to τ, where $\hat{d}_{ij}$ are bounded functions on $G(d)$. We put $\hat{C}_0 := (\hat{b}_{ij})_{1\leq i,j\leq m}$ and $\hat{C}_2 := (\hat{d}_{ij})_{1\leq i,j\leq m}$. Then we have $A\hat{C}_1(p) + \hat{C}_0(p) + \hat{C}_2(p) > 0$ for $p \in G(d)$ if A is sufficiently large. $\hat{M}$ denotes the matrix representation of the hermitian form $\sqrt{-1}\partial\overline{\partial}\mu + \sqrt{-1}\partial\overline{\partial}(A\sum|(\pi\circ\sigma)|^2) + \sqrt{-1}\partial\overline{\partial}B\widetilde{\varphi}$ with respect to $\hat{\tau}$. Then we have $|\hat{M}| = (\hat{b}_{m+1} + \hat{d}_{m+1m+1})|A\hat{C}_1| + \hat{Q}_{m-1}(A)$,where $\hat{Q}_{m-1}(A)$ denotes the $(m-1)$–degree polynomial with respect to A whose coefficients are bounded functions on $G(d)$. There is a constant $a_3 > 0$ satisfying that $\hat{b}_{m+1}(p) + \hat{d}_{m+1m+1}(p) > a_3$ for $p \in G(d)$ since $\sqrt{-1}\partial\overline{\partial}\Psi|_{\hat{l}(p)} > 0$ holds for $p \in G(t)$. Hence we have $|\hat{M}(p)| > 0$ if A is sufficiently large. Then we have $\sqrt{-1}\partial\overline{\partial}\Psi(p) > 0$ on $\hat{H}(p)$ for $p \in G(d)$ if A is sufficiently large.We fix such a constant $A =: A_2$. We put $A := \max\{A_1, A_2\}$. In view of minimum–maximum principle(cf :[C–H]), Ψ is a strongly n–convex exhaustion function on $G = \widetilde{S}(D(t), U(t))$ for constants A, B.

Case 2: Let $t \in (0,\infty)$ be a critical value of h. Let $y \in \widetilde{X}_0$ be a critical value of h such that $h(y) = t$. Then there is a neighborhood $\widetilde{W_j} \subset \widetilde{X}_0$ of y satisfying followings: (i) $\widetilde{r}|_{\widetilde{S}(\widetilde{W}_j,U)}$ is a holomorphic retraction (ii) $\widetilde{h}$ is strongly n–convex on $\widetilde{S}(\widetilde{W_j}, U) \cap \widetilde{X}_z$ for $z \in U$. We put $R(y) := \widetilde{S}(\widetilde{W_j}, U) \cap G(t)$.

Let $\{p_i \in Y(z)\}$ be a sequence of points such that $\{p_i\}$ converges to $p_0 \in \widetilde{S}(y, z)$ as $i \longrightarrow \infty$ for $z \in U(t)$. Then we have

$\triangle_g\mu|_{Y(z)}(p_i) \geq Trace_g\sqrt{-1}\left(\frac{\partial\overline{\partial}(\widetilde{h})}{c-\widetilde{h}}\right)\Big|_{Y(z)}(p_i) \longrightarrow \infty$ as $i \longrightarrow \infty$ by the definition of the hermitian metric g on $\widetilde{X}_U$. Hence $\triangle_g\mu|_{Y((\pi\circ\sigma)(p))}$ is bounded from below on $G(t)$. So Ψ is strongly g–subharmonic on each $Y(z)$ for $z \in U(t)$ if B is sufficiently large. We fix such a positive constant B.

Let p be any point of $R(y)$. Then there is an 1–dimensional complex subspace $l^*(p) \subset T^{1,0}_pY((\pi\circ\sigma)(p))$ satisfying that $\partial\overline{\partial}\Psi|_{l^*(p)} > 0$. We put an $(m+1)$–dimensional complex subspace $H^*(p) := T^{1,0}_pZ(p) \oplus l^*(p) \subset T^{1,0}_pG(t)$. Let $\tau^*(p) = (\tau^*_1(p), \cdots, \tau^*_{m+1}(p))$ be a normal basis of $H^*(p)$ with respect to g satisfying that $span\langle\tau^*_1(p), \cdots, \tau^*_m(p)\rangle_{\mathbf{C}} = T^{1,0}_pZ(p)$,$span\langle\tau^*_{m+1}(p)\rangle_{\mathbf{C}} = l^*(p)$ for $p \in R(y)$ and τ^*_i are continuous sections in $T^{1,0}R(y)$ for $i = 1, \cdots, m+1$. Then we have

$$\sqrt{-1}\partial\overline{\partial}\mu =: \begin{pmatrix} 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & \frac{b_0^*}{(t-h)^2} + \frac{b_{m+1}^*}{(t-h)} \end{pmatrix}$$

with respect to τ^*, where b_0^* is a positive continuous function and b_{m+1}^* is a continuous functions such that there is a constant $a_4 > 0$ satisfying that $b_{m+1}^*(p) > a_4$ for $p \in R(y)$. Indeed $\widetilde{r}$ is a holomorphic retraction from $\widetilde{S}(\widetilde{W}_j, U)$ to $\widetilde{W}_j$. We put $\sqrt{-1}\partial\overline{\partial}(A\sum|(\pi\circ\sigma)|^2) =: A(c_{ij}^*)_{1\leq i,j\leq m+1}$,where c_{ij}^* are bounded functions on $R(y)$. we put $C_1^* := (c_{ij}^*)_{1\leq i,j,\leq m}$. Then there is a constant $a_5 > 0$ satisfying that $C_1^* > a_5 I_m$ on $G(b,t)$. We set $\sqrt{-1}\partial\overline{\partial}B\widetilde{\varphi} =: (d_{ij}^*)_{1\leq i,j\leq m+1}$ with respect to τ^*, where d_{ij}^* are bounded functions on $R(y)$. We put $C_2^* := (d_{ij}^*)_{1\leq i,j\leq m}$. Then we have $AC_1^* + C_2^* > 0$ for $p \in R(y)$ if A is sufficiently large. M^* denotes the matrix representation of the hermitian form $\sqrt{-1}\partial\overline{\partial}\mu + \sqrt{-1}\partial\overline{\partial}(A\sum|(\pi\circ\sigma)|^2) + \sqrt{-1}\partial\overline{\partial}B\widetilde{\varphi}$ with respect to τ^*. Then we have $|M^*| = \{\frac{b_0^*}{(t-h)^2} + \frac{b_{m+1}^*}{t-h}\}(A^m|C_1^*| + Q_{m-1}^*(A)) + Q_{m+1}^*(A)$,where $Q_i^*(A)$ denotes the i–degree polynomial with respect to A whose coefficients are bounded functions on $R(y)$. Let $\{q_i \in G(t)\}$ be a sequence such that $\{q_i\}$ converges to a point $q_0 \in \widetilde{S}(y, \overline{U(t)})$. We have $|M(q_i)| \longrightarrow +\infty$ as $i \longrightarrow +\infty$ if A is sufficiently large. Hence there is a neighborhood $\widetilde{W}_j^* \subset \widetilde{X}_0$ of y satisfying that $\widetilde{W}_j^* \subset \widetilde{W}_j$ and $|M^*(p)| > 0$ for any $p \in \widetilde{S}(\widetilde{W}_j^*, U) \cap G(t)$. Then we find $\sqrt{-1}\partial\overline{\partial}\Psi(p) \geq \sqrt{-1}\{\partial\overline{\partial}\mu(p) + \partial\overline{\partial}(A\sum|(\pi\circ\sigma)(p)|^2) + \partial\overline{\partial}B\widetilde{\varphi}(p)\} > 0$ on $H(p)$ for any $p \in \widetilde{S}(\widetilde{W}_j^*, U) \cap G(t)$ if A is sufficiently large.We fix such a constant $A =: A_3$.

Let p be any point of $G(t) \setminus \widetilde{S}(\widetilde{W}_j^*, U)$. Then we find that there is a sufficently large constants A_4 and an $(m+1)$–dimensional complex subspace $H(p) \subset T_p^{1,0}G(t)$ satisfying that $\sqrt{-1}\partial\overline{\partial}\Psi(p) > 0$ on $H(p)$ by the same argument in Case 1. We put $A = \max\{A_3, A_4\}$. Then $\Psi(p)$ is a strongly n–convex exhaustion function on $G(t)$ for constants A, B.Thus we have Proposition3.5.

4. Construction of an exhaustion sequence of n–Runge pairs on $\widetilde{X}_U$

Let $\pi : X \longrightarrow T, \sigma : \widetilde{X} \longrightarrow X$ be as above. We have fixed C^∞–maps $\widetilde{S}$, $\widetilde{r}$ and $\{\widetilde{W}_j \subset \widetilde{X}_0\}_{j\in\mathbf{N}}$ as in Theorem2.3 with respect to π , σ and $U = \{z \in \mathbf{C}||z| < 1\}$

,by altering the coorninate of T if necessary. Let $h : \widetilde{X}_0 \longrightarrow [0,\infty)$ be a Morse exhaustion function satisfying Proposition3.4. We set $G(t) := \widetilde{S}(D(t), U(t))$ for $t \in (0,\infty)$. $\{G(t) | t \in (0,\infty)\}$ is a proper increasing continuous family of $\widetilde{X}_U$ satisfying that $\widetilde{X}_U = \bigcup_{t\in(0,\infty)} G(t)$. Hence we have our claim from Theorem of Docquier–Grauert and Proposition3.5 for $n = 1$ (cf:[D–G],[Oh2]). In this section we will show that our claim holds even if $n > 1$.

(1) Runge pairs of the family of relatively compact domains

Let $s \in (1,\infty)$ be any constant. We consider the family $\{G(t)\}$ on the finite interval $I := \{t \in [1, s]\}$. Let $\widetilde{\varphi}$ be a bounded function on $G(s)$ such that $\widetilde{\varphi}$ is strongly g–subharmonic on each n–dimensional complex submanifold $G(s) \cap \widetilde{X}_z$ at $z \in U(s)$ in Proposition3.3.

We set $\mu_t(p) := -\log(t - (h \circ \widetilde{r})(p))$, $\nu_t(p) := -\log(1 + \exp(-t) - |(\pi \circ \sigma)(p)|^2)$ and $\Psi[t](p) := \mu_t(p) + \nu_t(p) + A|(\pi \circ \sigma)(p)|^2 + B\widetilde{\varphi}(p)$ for $t \in [1, s]$ and $p \in G(t)$, where A and B are positive constants.Then we have the following.

Proposition4.1. *There are large constants A, B such that $\Psi[t]$ is a strongly n–convex exhasution function on $G(t)$ for $t \in I$,where A, B are independent of $t \in I$.*

Proof: $\widetilde{\varphi}$ is bounded and $\{G(t)\}$ is a continuous family. Hence we have such constants A, B in the same way to Proposition3.5.

We may assume that $\inf_{p\in G(t)} \{A\ |(\pi \circ \sigma)(p)|^2 + B\widetilde{\varphi}(p)\} = 0$ for any $t \in I$ by the construction of $\widetilde{\varphi}$. We put $\alpha := \sup_{t\in I, p\in G(t)} \{A|(\pi \circ \sigma)(p)|^2 + B\widetilde{\varphi}(p)\}$ and $\beta := \inf_{t\in I, p\in G(t)} \{\mu_t(p), \nu_t(p)\}$. Then the following holds.

Lemma4.2. *(1)Let $a \in [1, s)$ be a constant and δ be a positive number. Then there is a constant $b \in (a, s]$ and a positive number ε satisfying the following conditions: (i) $N := \{p \in G(b) | \mu_b(p) > -\log\varepsilon$ or $\nu_b(p) > -\log\varepsilon\} \supset G(b) \setminus G(a)$ (ii) $\varepsilon < \frac{1}{4}\exp(\beta - \alpha) \cdot \delta^2$ (2) Let $a \in [1, s)$, $b \in (a, s)$, $\varepsilon > 0$, $\delta > 0$ be constants satisfying conditions of (1). Then there is a constant $C \in \mathbf{R}$ satisfying that $G(a)$ contains $L := \{p \in G(b) | \Psi[b](p) < C\}$ and L contains $M^* := \{p \in G(b) | \mu_b(p) < -\log\frac{\delta}{2}, \nu_b(p) < -\log\frac{\delta}{2}\}$.*

Proof : (1)We put $d(u) := \inf\{\max\{\mu_u(p), \nu_u(p)\} | p \in G(u) \setminus G(a)\}$ for $u \in (t, \infty)$.Then we have $d(u) \to +\infty$ as $u \to t$. Hence such constants b, ε exist.

(2) By the conditions of $a, b, \varepsilon, \delta$ we have the following.

$$\Psi[b](p) < -2\log\frac{\delta}{2} + A \quad \text{on} \quad M^* \tag{$*$}$$

On the other hand we have the following.

$$\Psi[b](p) > -\log\varepsilon + \alpha \quad \text{on} \quad N \tag{$**$}$$

We put $C := -2\log\frac{\delta}{2} + A$ and $L := \{p \in G(b) | \Psi[b](p) < C\}$. Then we find that L contains M^* by $(*)$. Moreover L and N do not have intersections by $(*), (**)$ and the condition (ii). Hence $G(b) \setminus G(a)$ and L do not have intersections. So $G(a)$ contains L. Thus we have Lemma4.2.

By using the argument of Lemma4.2 we can interpolate between $G(u)$ and $G(v)$ for $u, v \in (1, s)$ by a part of an exhaustion sequence of n–Runge pairs as following.

Proposition 4.3. *Let δ be a positive number. Let $u, v \in (1, s)$ be constants with $u < v$. Then there is a number $J \in \mathbf{N}$,a sequence of numbers $\{t(0) < t(1) < \cdots < t(J) | t(0) = u, t(J) = v\}$, a positive constant C satisfying that $G(t(j))$ contains $L(j) := \{p \in G(t(j+1)) | \Psi[t(j+1)](p) < C\}$ and $L(j)$ contains $M^*(j) := \{p \in G(t(j+1)) | \mu_{t(j+1)}(p) < -\log\frac{\delta}{2}, \nu_{t(j+1)}(p) < -\log\frac{\delta}{2}\}$ for $0 \le j < J$.*

Proof: For $t \in [1, s)$, we put

$\varepsilon^*(t) := \sup\Big\{0 < \rho < \frac{1}{4}\exp(\beta - \alpha)\delta^2 \Big|$ there is $u \in (t, \infty)$ such that $N(u, \rho) \supset G(u) \setminus G(t)\Big\}$, where we put $N(u, \rho) := \{p \in G(u) | \mu_u(p) > -\log\rho$ or $\nu_u(p) > -\log\rho\}$ for $u \in (t, \infty)$ and $\rho \in (0, \infty)$. Then we have $\varepsilon^*(t) > 0$ for $t \in [1, s)$ by Lemma4.2(1). $\varepsilon^*(t)$ is continuous on $[1, s)$ since $\{G(t)\}$ is a continuous family. We put $\varepsilon := \min_{t \in [u,v]}\{\varepsilon^*(t)\}$. Then there is a sequence $\{t(0) < t(1) < \cdots < t(J) | t(0) = u, t(J) = v\}$ such that $\varepsilon < \frac{1}{4}\exp(\alpha - A) \cdot \delta^2$ holds and $N_j := \{p \in G(t(j+1)) | \mu_{t(j+1)}(p) > -\log\varepsilon$ or $\nu_{t(j+1)}(p) > -\log\varepsilon\}$ contains $G(t(j+1)) \setminus G(t(j))$ for $0 \le j < J$ by the construction of ε. We set $C := -2\log\frac{\delta}{2} + A$, $a := t(j)$ and

$b := t(j+1)$ for $0 \leq j < J$. Then we apply Lemma4.2(2). We set $L(j) := L$ and $M^*(j) := M$ for $a = t(j), b = t(j+1)$ in Lemma4.2(2). Then we have Proposition4.3.

(2) Proof of Theorem

Let $\{s_i \in (1,\infty) | i \in \mathbf{N}\}$ be an increasing sequence of numbers which diverges to $+\infty$. We put $s_0 = 1$. By Proposition3.3,there is a bounded function $\widetilde{\varphi}_i$ on $G(s_i)$ for $i \in \mathbf{N}$ such that $\widetilde{\varphi}_i$ is strongly g–subharmonic on each n–dimensional complex submanifold $G(s_i) \cap \widetilde{X}_z$ for $z \in U(s_i)$.

Let $\mu_t(p), \nu_t(p)$ be as above for $t \in (0,\infty)$. We put $\Psi_i[t](p) := \mu_t(p) + \nu_t(p) + A_i|(\pi \circ \sigma)(p)|^2 + B_i\widetilde{\varphi}_{i+1}(p)$,where A_i, B_i are positive constants. $\Psi_i[t]$ is a strongly n–convex exhaustion function on $G(t)$ for any $t \in [1, s_{i+1}]$ by Proposition4.1 if A_i, B_i are sufficiently large. We may assume $\inf_{p \in G(s_{i+1})} \{A_i|(\pi \circ \sigma)(p)|^2 + B_i\widetilde{\varphi}_{i+1}(p)\} = 0$. Let $\{\delta_i > 0\}_{i \in \mathbf{N}}$ be a decreasing sequence of positive numbers.

We put $u = s_{i-1}, v = s_i, \delta = \delta_i$ and apply Proposition4.3.Then we have $J_i := J \in \mathbf{N}, C_i := C \in \mathbf{R}, \{t(i,j)\} := \{t(j) \in [s_{i-1}, s_i] | t(j) < t(j+1), t(0) = s_{i-1}, t(J) = s_i\}, L(i,j) := L(j), M^*(i,j) = M^*(j) \subset \widetilde{X}_U$ satisfying that (i) $G(t(i,j)) \supset L(i,j) := \{p \in G(t(i,j+1)) | \Psi[t(i,j+1)](p) < C_i\}$ (ii) $L(i,j) \supset M^*(i,j) := \{p \in G(t(i,j+1)) | \mu_{t(i,j+1)}(p) < -\log \frac{\delta_i}{2}, \nu_{t(i,j+1)}(p) < -\log \frac{\delta_i}{2}\}$ for $i \in \mathbf{N}, 0 \leq j < J_i$.

We put $k := k(i,j) = \sum_{a=1}^{i-1} J_a + j + 1$ for $i \in \mathbf{N}$, $0 \leq j < J_i$. We set $G_k := G(t(i,j)), \Psi_k := \Psi_i[t(i,j)]$ and $L_k := L(t(i,j)), M_k^* := M(t(i,j))$. Then $\{(G_k, \Psi_k)\}$ is an exhaustion sequence of n–Runge pairs on $\widetilde{X}_U$. Indeed followings hold. L_k is the sublevel set of G_{k+1} and $G_k \supset M_k^* \supset L_k$ holds for any $k \in \mathbf{N}$. Let $\{M_k \subset M_k^*\}_{k \in \mathbf{N}}$ be an increasing sequence of subsets of $\widetilde{X}_U$ satisfying that M_k is a compact subset of L_k such that M_k contains M_{k-1}^* for any $k \in \mathbf{N}$, where we put $M_0 = \phi$. Then we have $\widetilde{X}_U = \bigcup_{k \in \mathbf{N}} M_k$ since $\{\delta_i > 0\}$ is a decreasing sequence. M_k is a compact subset of L_k and L_k is the sublevel set of G_{k+1} which is contained in G_k for $k \in \mathbf{N}$. Hence we have $H^i(\widetilde{X}_U, \mathcal{F}) = 0$ if $r \geq n$ for any coherent analytic sheaf $\mathcal{F}$ over $\widetilde{X}_U$ from Proposition 1.2.

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